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A GENERALIZATION OF KAEHLER GEOMETRY

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1. Introduction

In this paper a class of non-Kaehler manifolds is introduced which by its very definition is included in the generalization of Kaehler geometry given by Chern [1] (see also Weil [8]). This class is of particular interest because of its additional structure thereby yielding in the compact case topological consequences of special interest. The spaces considered are the globally framed f-manifolds $M(f, E_a, g)$, $a = 1, \cdots, 2n - r$, where $\dim M = 2n$ is even and rank $f = r$, previously studied by Yano and the author in [2]–[4]. Thus, it is necessary that the structural group of the tangent bundle of M can be reduced to thc direct product of $U(r/2)$ and $O(m - r)$, the unitary group in $r/2$ complex variables and the orthogonal group in $m - r$ variables. In [3], the structure tensors f and the E_a are assumed to be parallel fields with respect to the Riemannian connection, but since this implies that there is an underlying Kaehlerian structure the theory is not a satisfactory one. The proper generalization along these lines is provided by assuming (a) the fundamental form F of the f-structure is closed, (b) the Nijenhuis torsion of f vanishes, and (c) the field f is parallel along the integral curves of the vector fields E_a. Conditions (a)–(c) are clearly satisfied if (a) is replaced by the stronger condition that f be a parallel field and, in fact, they are equivalent to the latter (Theorem 1, Corollary 2). When $r = m$, the f-structure of M is Kaehlerian.

Chern's generalization of Kaehlerian geometry may be described as follows. Suppose that the structure group of the tangent bundle of a real C^∞ manifold of dimension m is reducible to a subgroup G of the rotation group in m variables. (Observe that $U(r/2) \times O(m - r) \subset O(m)$.) A connection can be defined with the group G. The vanishing of torsion of this connection is then a natural generalization of the Kaehler property. This includes the generalization due to Lichnerowicz [6], namely the even dimensional orientable Riemannian manifolds carrying a 2-form, of maximal rank everywhere, whose covariant derivative vanishes.

Conditions (a) and (b) are analogous to those characterizing Kaehler manifolds, whereas (c) is required when the rank of f is less than $2n$, and otherwise is vacuous. The f-manifold has an associated Kaehler structure if and only if

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the $2n - r$ pfaffian forms $\eta^a = g(E_a, \cdot)$ are closed. If f is everywhere of highest rank, then F is the Kaehler form. The theory of harmonic differential forms is employed to obtain the cohomology of these spaces, and a decomposition theorem generalizing the one obtained by Hodge for compact Kaehler manifolds is given, the invariant r playing a significant role.

There is also an obvious odd dimensional generalization provided by those framed manifolds satisfying conditions (a)–(c).

2. Framed manifolds

A Kaehler manifold is an hermitian manifold which is symplectic for the fundamental 2-form Ω of the hermitian structure. That Ω is then a parallel field is a consequence of the integrability of its almost complex structure J, that is, its Nijenhuis torsion $[J, J]$ vanishes, where $[J, J](X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$.

An m-dimensional C^∞ manifold M which carries a linear transformation field $f \neq 0$ of class C^∞ satisfying the algebraic condition $f^3 + f = 0$ is called an *f-manifold* provided the *f-structure* f is of constant rank r on M. Such structures exist if the structural group of the tangent bundle of M is reducible to $U(r/2) \times O(m - r)$, and conversely. Observe that r is even. As examples there are the almost complex structures for $m = 2n$ and the almost contact structures for $m = 2n - 1$, the former having maximal rank and the latter having rank $2n - 2$.

By putting

$$s = -f^2\,, \qquad t = f^2 + I\,,$$

where I is the identity transformation field, we have

$$s + t = I\,, \quad s^2 = s\,, \quad t^2 = t\,, \quad f^2 s = -s\,, \quad ft = 0\,.$$

The operators s and t acting in the tangent space at each point of M are therefore complementary projection operators defining distributions S and T in M corresponding to s and t, respectively. The distribution S is r-dimensional and $\dim T = m - r$.

If there are $m - r$ vector fields E_a spanning T at each point of M, and $m - r$ pfaffian forms η^a satisfying

$$\eta^a(E_b) = \delta^a_b\,, \tag{2.1}$$

where δ^a_b, $a, b = 1, \cdots, m - r$, is the 'Kronecker delta', and if the structure tensors are related by

$$f^2 = -I + \eta^a \otimes E_a\,, \tag{2.2}$$

where $\otimes$ denotes the tensor product, then M is said to be a *globally framed* *f-manifold* or, simply, a *framed manifold*; the summation convention is employed here and occasionally in the sequel.

As examples, there are the almost complex manifolds for $m = 2n$ and the almost contact spaces for $m = 2n - 1$. (Strictly speaking, because of the former example, the indices a, b should run though $0, 1, \cdots, m - r$ with $E_0 = 0$ and $\eta^0 = 0$.) The framed structure on M will be denoted by $M(f, E_a, \eta^a)$. From (2.1) and (2.2), one easily obtains

$$fE_a = 0 \;, \quad \eta^a \circ f = 0 \;, \qquad a = 1, \cdots, m - r \;. \tag{2.3}$$

The framed manifold $M(f, E_a, \eta^a)$, $a = 1, \cdots, m - r$, is called a *framed metric manifold* if a Riemannian metric g on M is distinguished such that

(i) $$\eta^a = g(E_a, \cdot) \;, \qquad a = 1, \cdots, m - r \;,$$

(ii) $$g(fX, Y) = -g(X, fY) \;.$$

Note that (ii) implies that f is skew-symmetric with respect to g, and (i) that the E_a form an orthonormal basis at each point of T. A framed manifold carries many metrics with these properties. We put

$$F(X, Y) = g(fX, Y)$$

and call F the *fundamental* 2-*form* of the framed structure.

Observing that on a framed manifold of any rank r

$$\iota(E_a)F^{r/2} = 0$$

for each $a = 1, \cdots, m - r$, and therefore

$$\frac{1}{m - r} \sum_{a=1}^{m-r} \iota(E_a)\varepsilon(\eta^a)F^{r/2} = F^{r/2} \;,$$

where ι and ε are the interior and exterior product operators, respectively. Denoting by $*$ the Hodge star operator, we see that

$$*F^{r/2} = k\eta^1 \wedge \cdots \wedge \eta^{m-r} \;,$$

where k is the C^∞ function given by $\pm\iota(E_1)\cdots\iota(E_{m-r}) * F^{r/2}$. Since

$$|F^{r/2}|^2 * 1 = F^{r/2} \wedge *F^{r/2} = k\eta^1 \wedge \cdots \wedge \eta^{m-r} \wedge F^{r/2} \;,$$

$$\eta^1 \wedge \cdots \wedge \eta^{m-r} = \pm\frac{1}{|F^{r/2}|} *F^{r/2} \;,$$

from which

$$*1 = \pm \frac{1}{|F^{r/2}|}\eta^1 \wedge \cdots \wedge \eta^{m-r} \wedge F^{r/2} ,$$

a formula giving the volume element of (M, g).

Let $M(f, E_a, \eta^a)$ be a framed metric manifold of dimension $m = 2n$ and rank r. Then, an almost complex structure

$$\tilde{f} = f + \eta^{2i} \otimes E_{2i-1} - \eta^{2i-1} \otimes E_{2i} ,$$

$i = 1, \cdots, n - r/2$, is defined on M in terms of which the metric g is hermitian. It follows that a framed manifold is orientable, a fact required in § 4. (If $\dim M = 2n + 1$, an almost contact metric structure $(\tilde{f}, E_{2n-r+1}, \eta^{2n-r+1})$ is defined.) Setting $\tilde{F}(X, Y) = g(\tilde{f}X, Y)$, we obtain

$$\tilde{F} = F + 2 \sum_i \eta^{2i} \wedge \eta^{2i-1} , \qquad i = 1, \cdots, n - r/2 . \tag{2.4}$$

If the fundamental form F and the η^a are closed, the almost hermitian structure $(\tilde{f}, g)$ on M is almost Kaehlerian. It is Kaehlerian if either $\tilde{f}$ has vanishing covariant derivative, or by Theorem 1 of [4], $M(f, E_a, \eta^a)$ is normal, that is, $[f, f] + d\eta^a \otimes E_a = 0$. (In this case, the E_a are holomorphic vector fields with respect to $\tilde{f}$.) By (2.4), $\tilde{f}$ is parallel if f and the η^a are also parallel fields, that is, $M(f, E_a, \eta^a)$ is a K-manifold (see [3]). Thus a K-manifold carries a Kaehler structure.

3. Quasi-symplectic manifolds

An even dimensional framed metric manifold $M(f, E_a, g)$ of rank r is called *quasi-symplectic* if F is closed and parallel along the integral curves of the vector fields E_a (see [3]). It is symplectic if $\dim M = 2n$ and $r = 2n$. We shall be primarily concerned with compact even dimensional quasi-symplectic spaces of rank less than $2n$. If, in addition the torsion $[f, f]$ is zero, a theory on M analogous to Weil's generalization of Hodge's theory on algebraic varieties may be developed. Under these conditions we shall see that $D_X F$ vanishes if X is *horizontal*, that is, if for each $P \in M$, $X(P)$ is orthogonal (with respect to g) to the subspace spanned by the $E_a(P)$, $a = 1, \cdots, 2n - r$, where D_X is the operator denoting covariant differentiation with respect to the Riemannian connection. Thus f is a parallel field. A generalization of Kaehler geometry is thereby obtained, since M is endowed with a Kaehler structure if and only if the η^a are closed forms.

A quasi-symplectic manifold with zero torsion will be called an *integrable quasi-symplectic manifold*.

Theorem 1. *Let $M(f, E_a, \eta^a)$ be a framed metric f-manifold with zero torsion. If the fundamental 2-form of M is closed, then*

$$D_X f = \eta^\alpha(X) D_{E_\alpha} f \tag{3.1}$$

for any vector field X on M.

Corollary 1. *If X is a horizontal vector field, then* $D_X f = 0$.

Corollary 2. *The linear transformation field f of an integrable quasi-symplectic manifold is a parallel field.*

Proof. Evaluating the torsion in terms of covariant derivatives we get

$$\begin{aligned}[f, f](X, Y) &= [fX, fY] - f[X, fY] - f[fX, Y] + f^2[X, Y] \\ &= D_{fX}(fY) - D_{fY}(fX) - f\{D_X(fY) - D_{fY}X\} \\ &\quad + f\{D_Y(fX) - D_{fX}Y\} - D_X Y + D_Y X + \eta^\alpha([X, Y])E_\alpha \\ &= (D_{fX}f)Y - (D_{fY}f)X - f\{(D_X f)Y - (D_Y f)X\} \\ &\quad - \eta^\alpha(D_X Y)E_\alpha + \eta^\alpha(D_Y X)E_\alpha + \eta^\alpha([X, Y])E_\alpha .\end{aligned}$$

Hence

$$\begin{aligned}&g((D_{fX}f)Y, Z) - g((D_{fY}f)X, Z) \\ &\quad - g((D_Y f)X, fZ) + g((D_X f)Y, fZ) = 0 .\end{aligned} \tag{3.2}$$

Evaluating the exterior derivative of F, we get

$$\begin{aligned}dF(X, Y, Z) &= X\cdot F(Y, Z) - Y\cdot F(X, Z) + Z\cdot F(X, Y) \\ &\quad - F([X, Y], Z) + F([X, Z], Y) - F([Y, Z], X) \\ &= g(D_X(fY), Z) + F(Y, D_X Z) + g(D_Y(fZ), X) \\ &\quad + F(Z, D_Y X) + g(D_Z(fX), Y) + F(X, D_Z Y) \\ &\quad - F([X, Y], Z) + F([X, Z], Y) - F([Y, Z], X) ,\end{aligned}$$

so that

$$g((D_X f)Y, Z) + g((D_Y f)Z, X) + g((D_Z f)X, Y) = 0 ,$$

since F is closed. Replacing Z by fZ in the last relation and subtracting from (3.2) we obtain

$$g((D_{fX}f)Y, Z) - g((D_{fY}f)X, Z) - g((D_{fZ}f)X, Y) = 0 ,$$

since $g((D_Z f)X, Y) + g((D_Z f)Y, X) = 0$. Interchanging Y and Z in the previous equation and substracting, $g((D_{fX}f)Y, Z) = 0$, from which it follows that $D_{fX}f = 0$ since g is definite. Applying (2.2) we get (3.1) and

$$(D_Z F)(X, Y) = \eta^\alpha(Z) g((D_{E_\alpha} f)X, Y) .$$

Theorem 1 is of fundamental importance for the study of the cohomology of integrable quasi-symplectic manifolds.

4. Cohomology of quasi-symplectic spaces

The most successful tool in the study of the homology of compact Kaehler manifolds is Hodge's theory of harmonic integrals [5] and [7]. We employ this method below. Define dual operators L and Λ on M of degrees 2 and -2 respectively by $L = \varepsilon(F)$ and $\Lambda = \iota(F)$. Then

$$\Lambda = (-1)^p * L *$$

on p-forms. A p-form ($p \geq 2$) is said to be *effective* if it is a zero of Λ. For $p = 0$ or 1 every form is said to be effective. On a framed metric manifold of rank $r < 2n$ there are many effective forms. Indeed, the exterior products $\eta^{i_1} \wedge \cdots \wedge \eta^{i_p}$ are effective p-forms. The notion of an effective form is a formulation in terms of cohomology of the effective cycles of Lefschetz on an algebraic manifold [5, p. 182].

An orthonormal basis of M_P of the form

$$\{X_A, X_{A^*}, E_a\}\ , \quad A = 1, \cdots, r/2\ , \quad X_{A^*} = fX_A\ , \quad a = 1, \cdots, 2n - r\ ,$$

$\dim M = 2n$, will be called an *f-basis*. To see that such a basis exists, let

$$M'_P = \{X \in M_P \,|\, g(X, E_a) = 0\ , \quad a = 1, \cdots, 2n - r\}\ .$$

Equations (2.1)–(2.3) show that $f|_{M'_P}$ is an almost complex structure on M'_P and $g|_{M'_P}$ is an hermitian metric. If an orthonormal (with respect to $g|_{M'_P}$) basis of M'_P of the form $\{X_A, (f|_{M'_P})X_A\}$, $A = 1, \cdots, r/2$, is then chosen, an f-basis of M_P is obtained.

In terms of an f-basis $\{X_A, X_{A^*}, E_a\}$ with dual basis $\{\omega_A, \omega_{A^*}, \eta^a\}$, L and Λ may be expressed as

$$L = \sum_{A=1}^{r/2} \varepsilon(\omega_A)\varepsilon(\omega_{A^*})\ , \qquad \Lambda = \sum_{A=1}^{r/2} \iota(X_{A^*})\iota(X_A)\ .$$

Since $\iota(X)$ is an anti-derivation, $\Lambda F = r/2$.

A p-form α on M is said to have *tridegree* (λ, μ, ν) if it is expressible as a sum of decomposable forms $\alpha = \omega_{A_1} \wedge \cdots \wedge \omega_{A_\lambda} \wedge \omega_{B_1^*} \wedge \cdots \wedge \omega_{B_\mu^*} \wedge \eta^{a_1} \wedge \cdots \wedge \eta^{a_\nu}$. We call $\alpha_h = \omega_{A_1} \wedge \cdots \wedge \omega_{A_\lambda} \wedge \omega_{B_1^*} \wedge \cdots \wedge \omega_{B_\mu^*}$ the *horizontal part* and $\alpha_v = \eta^{a_1} \wedge \cdots \eta^{a_\nu}$ the *vertical part*. Thus $\alpha = \alpha_h \wedge \alpha_v$. Clearly

$$(4.1) \qquad \Lambda\alpha = \Lambda\alpha_h \wedge \alpha_v\ .$$

Lemma 1. *On a framed metric manifold, L and Λ satisfy*

$$\Lambda L\alpha - L\Lambda\alpha = (r/2 + \nu - p)\alpha$$

for any p-form α *of tridegree* (λ, μ, ν).

Proof. By linearity, it suffices to consider the decomposable forms α_h and $\alpha_h \wedge \alpha_v$. The result then follows from formula (4.1) and the corresponding relation for almost hermitian spaces:

$$\begin{aligned}(\Lambda L - L\Lambda)\alpha &= \Lambda((L\alpha_h) \wedge \alpha_v) - L(\Lambda\alpha_h) \wedge \alpha_v \\ &= ((\Lambda L - L\Lambda)\alpha_h) \wedge \alpha_v = (r/2 - \lambda - \mu)\alpha_h \wedge \alpha_v \\ &= (r/2 + \nu - p)\alpha \ .\end{aligned}$$

We define the operators d', d'', d°, δ', δ'' and δ° in terms of the Riemannian connection of the framed metric structure

$$\begin{aligned} d' &= \sum_A \varepsilon(\omega_A)D_{X_A} \ , & d'' &= \sum_A \varepsilon(\omega_{A^*})D_{X_{A^*}} \ , & d^\circ &= \sum_A \varepsilon(\eta^a)D_{E_a} \ , \\ \delta' &= -\sum_A \iota(X_A)D_{X_{A^*}} \ , & \delta'' &= -\sum_A \iota(X_{A^*})D_{X_A} \ , & \delta^\circ &= -\sum_A \iota(E_a)D_{E_a} \ , \end{aligned}$$

$A = 1, \cdots, r/2$; $a = 1, \cdots, 2n-r$. Then the exterior differential operator d is the sum of d', d'' and d° and its dual δ is the sum of the duals δ', δ'' and δ° of d', d'' and d°. (Although the operators of exterior differentiation are defined explicitly in terms of the Riemannian metric g, only the property that the Riemannian connection is torsion free is relevant. Note also that the basis vectors are orthonormal with respect to g.) Observe that the primed operators have their analogues in almost hermitian manifolds.

Lemma 2. *On a framed manifold,*

$$\begin{aligned} d'd' &= 0 \ , & d'd'' + d''d' &= 0 \ , \\ d''d'' &= 0 \ , & d^\circ d' + d'd^\circ &= 0 \ , \\ d^\circ d^\circ &= 0 \ , & d^\circ d'' + d''d^\circ &= 0 \ . \end{aligned}$$

Proof. Since $dd = 0$, the relations follow by comparing tridegrees.

Lemma 3. *On an integrable quasi-symplectic manifold,*

(i) $\delta' L - L\delta' = -d''$,

(ii) $\delta'' L - L\delta'' = d'$,

(iii) $\delta^\circ L = L\delta^\circ$,

(iv) $\delta L - L\delta = d' - d''$.

Proof. Since F is closed, $D_X F = 0$ by (3.1), provided X is horizontal. Thus

$$\begin{aligned}\delta' L\alpha - L\delta'\alpha &= -\sum_A \iota(\omega_A)D_{X_{A^*}}(F \wedge \alpha) + F \wedge \sum_A \iota(\omega_A)D_{X_{A^*}}\alpha \\ &= -\sum_A \iota(\omega_A)(F \wedge D_{X_{A^*}}\alpha) + F \wedge \sum_A \iota(\omega_A)D_{X_{A^*}}\alpha\end{aligned}$$

$$= -\sum_A \varepsilon(\omega_{A*})D_{X_{A*}}\alpha - L\sum_A \iota(\omega_A)D_{X_{A*}}\alpha + L\sum_A \iota(\omega_A)D_{X_{A*}}\alpha$$
$$= -d''\alpha .$$

A similar computation gives (ii).

To obtain (iii), we use the fact that $D_{E_a}F = 0$, $a = 1, \cdots, 2n - r$. Then

$$\delta^\circ L\alpha = -\sum_a \iota(E_a)D_{E_a}(F \wedge \alpha) = -\sum_a L\iota(E_a)D_{E_a}\alpha = L\delta^\circ\alpha .$$

To obtain (iv) one simply adds (i), (ii) and (iii).

Lemma 4. *On a quasi-symplectic manifold*

(i) $dL = Ld$, $\quad\Lambda\delta = \delta\Lambda$,

(ii) $d'L = Ld'$, $\quad d''L = Ld''$, $\quad d^\circ L = Ld^\circ$,

(iii) $\delta'\Lambda = \Lambda\delta'$, $\quad\delta''\Lambda = \Lambda\delta''$, $\quad\delta^\circ\Lambda = \Lambda\delta^\circ$.

Proof. (i) is an immediate consequence of the fact that F is closed, and (ii) is obtained from it by comparing tridegrees. The relations (iii) are the duals of the corresponding formulas in (ii).

For the proof of Lemma 5 we shall require the dual of Lemma 2, namely the formulas

$$\delta'\delta' = 0 , \qquad \delta'\delta'' + \delta''\delta' = 0 ,$$
$$\delta''\delta'' = 0 , \qquad \delta^\circ\delta' + \delta'\delta^\circ = 0 ,$$
$$\delta^\circ\delta^\circ = 0 , \qquad \delta^\circ\delta'' + \delta''\delta^\circ = 0 .$$

Lemma 5. *On an integrable quasi-symplectic manifold,*

(i) $d'\delta'' + \delta''d' = 0$,

(ii) $d''\delta' + \delta'd'' = 0$,

(iii) $d'\delta^\circ + \delta^\circ d' = 0$,

(iv) $d''\delta^\circ + \delta^\circ d'' = 0$.

Proof. (i) and (iii) are both immediate from Lemma 3 and the dual of Lemma 2 as are (iii) and (iv). We give only the proof of (iii). By Lemma 3,

$$d'\delta^\circ = \delta''L\delta^\circ - L\delta''\delta^\circ = -\delta^\circ\delta''L + \delta^\circ L\delta'' ,$$
$$\delta^\circ d' = \delta^\circ\delta''L - \delta^\circ L\delta'' .$$

Adding these relations, we get (iii).

Lemma 6. *On an integrable quasi-symplectic manifold the Laplace-Beltrami operator Δ has the expressions*

$$\Delta = 2(d'\delta' + \delta'd') + (d^\circ\delta^\circ + \delta^\circ d^\circ) = 2(d''\delta'' + \delta''d'') + (d^\circ\delta^\circ + \delta^\circ d^\circ) .$$

Proof. Let Λ_h^p denote the linear space of horizontal p-forms. Then, from Lemma 3, the expression $\delta'L\delta'' + \delta''L\delta' - \delta''\delta'L + L\delta'\delta''$ is equal to

$d''\delta'' + \delta''d''$ from (i) and to $d'\delta' + \delta'd'$ from (ii). We need only show now that $\Delta = 2(d'\delta' + \delta'd') + (d^\circ\delta^\circ + \delta^\circ d^\circ)$, and to see this we expand $\Delta = d\delta + \delta d$:

$$\begin{aligned} d\delta + \delta d &= (d' + d'' + d^\circ)(\delta' + \delta'' + \delta^\circ) + (\delta' + \delta'' + \delta^\circ)(d' + d'' + d^\circ) \\ &= d'\delta' + d'\delta'' + d'\delta^\circ + d''\delta' + d''\delta'' + d''\delta^\circ \\ &\quad + d^\circ\delta' + d^\circ\delta'' + d^\circ\delta^\circ + \delta'd' + \delta'd'' + \delta'd^\circ \\ &\quad + \delta''d' + \delta''d'' + \delta''d^\circ + \delta^\circ d' + \delta^\circ d'' + \delta^\circ d^\circ \\ &= (d'\delta' + \delta'd') + (d''\delta'' + \delta''d'') + (d^\circ\delta^\circ + \delta^\circ d^\circ) \\ &= 2(d'\delta' + \delta'd') + (d^\circ\delta^\circ + \delta^\circ d^\circ) \end{aligned}$$

by Lemma 5.

Lemma 7. *On an integrable quasi-symplectic manifold, Δ commutes with L and Λ.*

Proof. Apply Lemmas 2–4. That $\Delta\Lambda = \Lambda\Delta$ follows from the fact that $*\Delta = \Delta*$.

As a matter of fact, the Laplace-Beltrami operator lies in the centre of the algebra of operators on an integrable quasi-symplectic manifold, and it is for this reason that Hodge theory is useful in obtaining the cohomology of these spaces.

Lemma 8. *On an integrable quasi-symplectic manifold M the forms $F^q = F \wedge \cdots \wedge F$ (q times) are harmonic of degree $2q$ for every integer $q \leq r/2$.*

The proof is by induction on the integer q. To begin with, F is harmonic. For, since M is quasi-symplectic, F is closed. Thus $d'F = 0$, $d''F = 0$ and $d^\circ F = 0$. By (i) of Lemma 3,

$$\delta'F = L\delta'1 - d''1 = 0.$$

Similarly, (ii) and (iii) yield

$$\delta''F = 0 \quad \text{and} \quad \delta^\circ F = 0 .$$

(That F is harmonic may also be seen by observing that F is a parallel tensor field.) Finally,

$$\Delta F^q = \Delta(LF^{q-1}) = L(\Delta F^{q-1}) = 0 .$$

Theorem 2. *The betti numbers $b_{2q}(M)$ of a compact integrable quasi-symplectic manifold M are different from zero for $q = 0, 1, \cdots, r/2$.*

Proof. The theorem is trivial for $q = 0$. The proof is now a consequence of the previous lemma and the fact that $F^q \neq 0$ for $q \leq r/2$. In fact, we need only show that $F^{r/2} \neq 0$, and this is so since $F^{r/2} \wedge \eta^1 \wedge \cdots \wedge \eta^{2n-r}$ defines an orientation of M.

5. Effective forms

There is a special class of forms defined as the zeros of the operator Λ on the space of harmonic p-forms. They are called *effective harmonic p-forms*, and the dimensions of the spaces determined by them are topological invariants. This important fact hinges on a relation measuring the defect of the operator $L^k\Lambda$ from ΛL^k where $L^k\alpha = \alpha \wedge F^k$. That these operators do not commute is crucial for the determination of these invariants.

Lemma 9. *On a framed metric manifold,*

$$(\Lambda L^k - L^k\Lambda)\alpha = k(r/2 + \nu - p - k + 1)L^{k-1}\alpha$$

for any p-form α of tridegree (λ, μ, ν), $p \leq r/2 + \nu - 2k + 2$.

Proof. By recursion on the integer k using Lemma 1:

$$\begin{aligned} \Lambda L^{k+1}\alpha &= \Lambda L^k(L\alpha) \\ &= L^k\Lambda(L\alpha) + k(r/2 + \nu - p - k - 1)L^k\alpha \\ &= L^k[L\Lambda\alpha + (r/2 + \nu - p)\alpha] + k(r/2 + \nu - p - k - 1)L^k\alpha \\ &= L^{k+1}\Lambda\alpha + (k + 1)(r/2 + \nu - p - k)L^k\alpha\ . \end{aligned}$$

Lemma 10. *If α is an effective p-form of tridegree (λ, μ, ν), then, for any integer $s \geq 0$,*

$$\begin{aligned} &(-1)^k\Lambda^kL^{k+s}\alpha \\ &\quad = (s + 1)\cdots(s + k)(s - n + p)\cdots(s - r/2 - \nu + p + k - 1)L^s\alpha\ . \end{aligned}$$

This follows inductively from the preceding lemma.

Corollary. *There are no effective p-forms of tridegree (λ, μ, ν) for $p \geq r/2 + \nu + 1$.*

This is an immediate consequence if one takes $k = r/2 + \nu + 1$ and $s \geq 0$.

Theorem 3. *On a framed metric manifold, a p-form α of tridegree (λ, μ, ν), $p \leq r/2 + \nu$, may be uniquely represented as a sum*

$$\alpha = \sum_{k=0}^{s} L^k\psi_{p-2k}\ , \tag{5.1}$$

where the ψ_{p-2k} are effective forms of degree $p - 2k$ and $s = [p/2]$.

Proof. The theorem is trivial for $p = 0$ and 1. Proceeding inductively, assume it is true for $p \leq r/2 + \nu - 2$. Then, associated to any p-form β, there is a unique p-form α such that

$$\Lambda L\alpha = \beta\ , \qquad p \leq r/2 + \nu - 2\ . \tag{5.2}$$

For,

$$\beta = \sum_{k=0}^{s} L^k \theta_{p-2k} ,$$

where the θ_{p-2k} are effective $(p-2k)$-forms. By (5.2) and Lemma 9,

$$\Lambda L\alpha = \sum_{k=0}^{s} \Lambda L^{k+1}\psi_{p-2k} = \sum_{k=0}^{s} (k+1)(r/2+\nu-p+k)L^k\psi_{p-2k} .$$

Since $p \leq r/2 + \nu - 2$, $r/2 - p + \nu + k \neq 0$, so in order that (5.2) hold, we need only take

$$\psi_{p-2k} = \frac{\theta_{p-2k}}{(k+1)(r/2+\nu-p+k)} , \qquad k = 0, 1, \cdots, s .$$

By uniqueness, this is also necessary. The remainder of the proof is omitted.

Denote by $\wedge^{\lambda,\mu,\nu}$ the linear space of p-forms of tridegree (λ, μ, ν).

Corollary 1. *On a framed metric manifold, ΛL is an automorphism of $\wedge^{\lambda,\mu,\nu}$ for $p \leq r/2 + \nu - 2$.*

Corollary 2. *On a framed metric manifold, L is an isomorphism of $\wedge^{\lambda,\mu,\nu}$ into $\wedge^{\lambda+1,\mu+1,\nu}$ for $p \leq r/2 + \nu - 2$.*

Assume now that M is an integrable quasi-symplectic manifold of rank r. Then, by Lemma 7, we obtain

Corollary 3. *On an integrable quasi-symplectic manifold, a harmonic p-form α of tridegree (λ, μ, ν), $p \leq r/2 + \nu$, may be uniquely represented as a sum*

$$\alpha = \sum_{k=0}^{s} L^k \psi_{p-2k} ,$$

where the ψ_{p-2k} are effective harmonic forms of degree $p-2k$ and $s = [p/2]$.

Corollary 4. *The betti numbers b_p of a compact integrable quasi-symplectic manifold satisfy the monotonicity condition $b_{p-2} \leq b_p$, $p \leq r/2$.*

For, L is an isomorphism sending harmonic $(p-2)$-forms into harmonic p-forms.

The difference $b_p - b_{p-2}$ may be measured in terms of the dimension e_p of the space of effective harmonic forms of degree p, $p \leq r/2$. For, by Corollary 3,

$$\wedge_H^p = \wedge_{He}^p \oplus L \wedge_{He}^{p-2} \oplus \cdots \oplus L^s \wedge_{He}^{p-2s} , \qquad s = [p/2] ,$$

where $\wedge_H^p$ and $\wedge_{He}^p$ denote the linear spaces of harmonic and effective harmonic p-forms, respectively. Hence

$$\wedge_H^{p+2} = \wedge_{He}^{p+2} \oplus L \wedge_H^p .$$

By Lemma 7 and Theorem 3, Corollary 2, $\dim L \wedge_H^p = \dim \wedge_H^p$, from which $b_{p+2} = e_{p+2} + b_p$, $p \leq r/2 - 1$.

Theorem 4. *On a compact integrable quasi-symplectic manifold,*

$$e_p = b_p - b_{p-2}\,, \qquad p \leq r/2\,.$$

Remarks. (a) If the η^a, $a = 1, \cdots, 2n - r$, are closed forms, then the effective forms $\eta^{i_1} \wedge \cdots \wedge \eta^{i_p}$ are harmonic forms of degree p. For, under these conditions, there is an underlying Kaehlerian structure given by $(\tilde{f}, g)$. In this case,

$$b_p \geq b_{p-2} + \binom{2n-r}{p}\,, \qquad p \leq r/2\,,$$

the parentheses denoting the binomial coefficient.

(b) Parallelisable manifolds are trivially quasi-symplectic and integrable since their fundamental forms vanish. The operator Δ is given by $d^\circ\delta^\circ + \delta^\circ d^\circ$. A p-form is expressible as a linear combination of the forms $\eta^{i_1} \wedge \cdots \wedge \eta^{i_p}$. Thus all forms are effective. If the η^a are closed, then M is Kaehlerian, and if M is compact, then it is a multi-torus.

6. Examples

Let N be a $(2n + 1)$-dimensional normal contact manifold with fundamental affine collineation φ, fundamental vector field E and contact form η. Consider a $2n$-dimensional manifold M imbedded in N with immersion $i\colon M \to N$ such that $E = i_* E'$. The structure induced on M turns out to be a framed structure of rank $2n - 2$ which is neither almost complex nor almost contact [3]. As examples, we may consider R^{2n} imbedded in R^{2n+1}, or the torus T^{2n} imbedded in T^{2n+1}. Let Φ be the fundamental 2-form of N. Then M is quasi-symplectic since $F = i^*\Phi$ is closed. Thus, if f is integrable, the framed structure on M is not normal, for then $i^*\eta$ would be closed and f would vanish (see [3, formulas (4.2)]). Observe that F is not a parallel field.

If the ambient space is a cosymplectic manifold, that is, if η is closed, then $\nabla\varphi = 0$ and $\nabla\eta = 0$, where ∇ denotes covariant differentiation with respect to the Riemannian connection of N. Denoting by D the induced connection on M, f is parallel with respect to D if M is totally geodesic. In this case, the framed structure on M is normal. Hence there is an underlying Kaehlerian structure. (There are no totally umbilical framed hypersurfaces of a normal contact manifold.)

To illustrate that our results transcend Kaehler geometry we need only take the direct product of a Kaehler manifold N and a parallelisable space P. (In the odd dimensional case, P may be the 3-sphere, for example.) This suggests the study of framed manifolds as bundle spaces over Kaehler manifolds with parallelisable fibres.

The deformation theory of framed manifolds is also suggested as a problem for future study. Indeed, families of Kaehler manifolds parametrized by a parallelisable space may be considered.

Bibliography

[1] S. S. Chern, *On a generalization of Kähler geometry,* Algebraic geometry and topology, A symposium in honor of S. Lefschetz, Princeton University Press, Princeton, 1957, 103–121.

[2] S. I. Goldberg, *Framed manifolds,* to appear in Differential Geometry in honor of Kentaro Yano, Kinokuniya, Tokyo, 1972.

[3] S. I. Goldberg & K. Yano, *Globally framed f-manifolds,* Illinois J. Math. **15** (1971) 456–474.

[4] ——, *On normal globally framed f-manifolds,* Tôhoku Math. J. **22** (1970) 362–370.

[5] W. V. D. Hodge, *The theory and application of harmonic integrals,* Cambridge University Press, Cambridge, 1941.

[6] A. Lichnerowicz, *Généralisations de la géométrie kählerienne globale,* Colloque de géométrie différentielle, Louvain, 1951, 99–122.

[7] A. Weil, *Sur la théorie des formes différentielles attachées à une variétés analytique complexe,* Comment. Math. Helv. **20** (1947) 110–116.

[8] ——, *Un théorème fondamental de Chern en géométrie riemannienne,* Séminaire Bourbaki, 14ième année, 1961–62, Exp. 239.

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